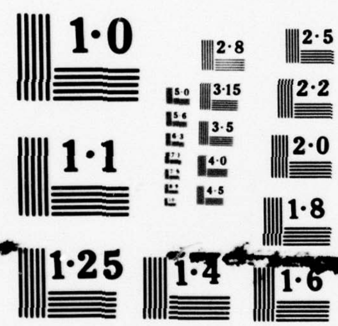
AD-A068 217 STANFORD UNIV CALIF DEPT OF OPERATIONS RESEARCH F/G 12/1
AN INFINITELY RETROGRESSING CONVERGING PATH IN F(-1)(0) DERIVED--ETC(U)
MAR 79 P S BROOKS DAAG29-78-G-0026
UNCLASSIFIED ARO-15254.5-M NL

1 OF 1
ADA
068217

END
DATE
FILMED
6-79
DDC



NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

6

AN INFINITELY RETROGRESSING CONVERGING PATH IN $F^{-1}(0)$

DERIVED FROM A C^{∞} FUNCTION AND THE J_3 TRIANGULATION

Infinity

by

10 Peter S. Brooks

15 DAAG29-78-G-0026, NSF-MCS77-05623

9 TECHNICAL REPORT,

March 10, 1979

11 10 Mar 79

12 33p.

18 ARO 19 15254.5-M

Department of Operations Research
Stanford University
Stanford, California

This research was sponsored in part by the Army Research Office - Durham, Contract No. DAAG-29-78-G-0026 and the National Science Foundation, Grant No. MCS-77-05623. Principal Investigator, B. Curtis Eaves.

ABSTRACT

For certain functions $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$, the Eaves-Saigal algorithm computes a path $p = (p_1, p_2) : [0, +\infty) \to F^{-1}(0) \cap \mathbb{R}^n \times (0,1]$, such that $(p_1(s), p_2(s)) \to (z,0)$ as $s \to +\infty$. It is shown that even when $F(\cdot,0)$ is of class C^∞ and has a unique *zero*, $p_2(s)$ *may not* decrease monotonically to 0 on $[s_0, +\infty)$ for any s_0.

AN INFINITELY RETROGRESSING CONVERGING PATH IN $F^{-1}(0)$ DERIVED FROM A C^∞ FUNCTION AND THE J_3 TRIANGULATION

Peter S. Brooks
Mathematics Department
Stanford University

Introduction

Triangulate $\mathbb{R}^n \times (0,1]$ with Todd's J_3 triangulation, the vertices of which are denoted by J^0 (Todd [10]). Elements of $\mathbb{R}^n \times [0,1]$ are written as (z,t). Given a map $f : \mathbb{R}^n \to \mathbb{R}^n$, we define a labeling $F : J^0 \to \mathbb{R}^n$ by $F((z,t)) = f(z)$. Extend F barycentrically on each simplex of the triangulation; call this new map F also. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, F may be extended from $\mathbb{R}^n \times (0,1]$ to $\mathbb{R}^n \times [0,1]$ in a natural manner by defining $F(z,0) = \lim_{t \to 0^+} F(z,t)$. We now observe that $f(z) = F(z,0)$. The Eaves-Saigal algorithm produces a piecewise linear 1-manifold which can be parametrized by $p = (p_1,p_2) : [0, +\infty) \to \mathbb{R}^n \times (0,1]$. This path is one path component of $F^{-1}(0) \cap \mathbb{R}^n \times (0,1]$ (Eaves and Saigal [6], Kojima [7]).

If $p_1(s)$ stays within some bounded region in $\mathbb{R}^n$, then for each $t < 1$, $p_2(s)$ eventually stays out of $[t,1]$, implying that $p_2(s) \to 0$ as $s \to +\infty$ (Eaves [4]). We say the path retrogresses on $[s',s'']$, $s' < s''$, if $p_2(s)$ is strictly increasing on $[s',s'']$, and if $p_2(s)$ fails to be strictly increasing on $[s' - \varepsilon, s'' + \varepsilon]$ for any $\varepsilon > 0$. If $p_2(s') = t'$, we say the path retrogresses at t'. See Figure 1. We say the path is infinitely retrogressing if there is a sequence $\{t^m\}$ monotonically decreasing to 0 such that the path retrogresses at each t^m. We say the path converges if $(p_1(s), p_2(s)) \to (z, 0)$ as $s \to +\infty$ for some z in $\mathbb{R}^n$.

Under what conditions might an infinitely retrogressing path occur? If the triangulated space is $\mathbb{R}^1 \times (0,1]$, then the algorithm is just the bisection algorithm (Eaves [4]). The bisection algorithm cannot yield any retrogressions at all.

If $p_1(s)$ is bounded in $\mathbb{R}^n$, then the cluster points of $p_1(s)$ form a non-empty closed connected set (Eaves [4]). Given that $f : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1, and that $\bar{z}$ is a cluster point of $p_1(s)$, Kojima proves that a non-vanishing Jacobian of f at $\bar{z}$ implies that the path cannot be infinitely retrogressing (Kojima [7]).

In this paper I shall show that infinitely retrogressing paths do occur. I shall construct a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying (i) f is of class C^∞, (ii) f has a unique zero at some z_0 in $\mathbb{R}^2$, (iii) $F^{-1}(0) \cap \mathbb{R}^2 \times (0,1]$ has an infinitely retrogressing converging path component.

Requirements for the Labeling

An n-simplex is the closed convex hull of $n + 1$ affinely independent points, called vertices. A j-dimensional face is the closed convex hull of any $j + 1$ of these vertices. An $(n-1)$-dimensional face is called a facet. Two facets of a triangulation are called adjacent if they intersect in an $(n-2)$-dimensional face. Note that any two facets of a given n-simplex are adjacent. In the following, we are using Todd's J_3 triangulation of $\mathbb{R}^2 \times (0,1]$. The simplices, in this case, are 3-simplices.

The first step in the construction of f is a "loose" specification of the value of f at certain points. Formally, given a set T in $\mathbb{R}^2$, and a point z in $\mathbb{R}^2$, we say z is prelabeled by T if $f(z)$ will be required to lie in T. If (z,t) is a vertex of the triangulation, and z is prelabeled by T, we also say (z,t) is prelabeled by T.

Let

$$X = \{(x,y) \in \mathbb{R}^2 : x > 0,\ y > 0\}$$

$$Y = \{(x,y) \in \mathbb{R}^2 : x < 0,\ y > 0\}$$

$$W = \{(x,y) \in \mathbb{R}^2 : x = 0,\ y < 0\}.$$

These sets are used to prelabel a specific sequence $\{z_n\}_{n=1}^{\infty}$. Figures 8-12 indicate the prelabeling of portion i of this sequence, namely $z_{9i+1}, \ldots, z_{9(i+1)+4}$ $(i = 0,1,2,\ldots)$. An X, Y, or W is placed beside each (z_n,t_k) according to whether (z_n,t_k) is prelabeled by X, Y, or W respectively. Each facet whose vertices are prelabeled by X, Y, and W is called completely prelabeled.

For a given map $f : \mathbb{R}^2 \to \mathbb{R}^2$, a facet τ is called completely f-labeled if $F(\tau)$ is a 2-simplex whose interior contains the origin. The key observation is that by the choice of the sets X, Y, and W, the completely prelabeled facets will become completely f-labeled once we have defined f on all of $\mathbb{R}^2$, subject to the prelabel requirements.

The idea underlying this approach is as follows: The simplex σ_1 in Figure 4 has exactly two completely prelabeled facets, τ_1 and τ_2.

No matter what values we assign to f at the z_{n_k}, subject to the prelabel requirements, there is exactly one point α_i in the interior of τ_i such that $F(\alpha_i) = 0$, $(i = 1,2)$. Since F is linear on σ_1, the chord between α_1 and α_2 is the preimage $F^{-1}(0)$ in σ_1. If σ_2 is the unique simplex which shares τ_2 with σ_1, and if the remaining vertex of σ_2 is prelabeled by X, Y, or W, then there are exactly two completely prelabeled facets, τ_2 and τ_3, of σ_2. As before, there is exactly one point α_3 in the interior of τ_3 such that the chord between α_2 and α_3 is the preimage $F^{-1}(0)$ in σ_2.

By a careful choice of prelabels for certain vertices, we form a sequence of completely prelabeled facets as indicated in Figures 8-12. This choice of prelabeling will result in a portion of $F^{-1}(0)$ which retrogresses once. We call this portion cycle i. Our construction enables us to piece together such portions, resulting in a path which retrogresses infinitely many times. We now describe this construction in detail.

The infinitely retrogressing path will begin at $p(0) = (p_1(0), p_2(0))$, with $p_1(0)$ in the interior of the square $\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$, and $p_2(0) = 1$. Cycle i is one cycle of the repeating pattern of the path. It is shown in detail in Figures 8-12. Figure 2 gives a schematic diagram of cycle i. Figure 3 indicates that portion of $\mathbb{R}^2 \times (0,1]$ used in the construction of cycle i. The successive stages of cycle i occur as follows:

1. The cycle begins at t_{3i} $(t_j = 2^{-j})$. The path passes down through block $B(i,1)$. Figure 8.

2. The path dips down into block $B(i,2)$ and returns to t_{3i+1}. It has retrogressed at t^i. Figure 9.

3. The path dips up into block $B(i,1)$ and returns to t_{3i+1}. Figure 10.

4. The path passes down through block $B(i,3)$. Figure 11.

5. The path passes down through block $B(i,4)$. At this point, we are at $t_{3(i+1)}$, ready to begin cycle $i + 1$. Figure 12.

The cycles are indexed by $i = 0, 1, 2, \ldots$. Cycle $i + 1$ has the properties that

1. the projection of $B(i+1,1)$ into $\mathbb{R}^2 \times \{1\}$ is contained in the interior of the projection of $B(i,1)$ into $\mathbb{R}^2 \times \{1\}$, and

2. the Y and W prelabels are interchanged in their positions relative to the X prelabels.

The next sections deal with the construction of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies the prelabel requirements on $\{z_n\}_{n=1}^{\infty}$. This function will automatically yield an infinitely retrogressing path which passes through an infinite sequence of distinct completely f-labeled facets $\{\tau_m\}$. The path intersects each τ_m at a unique point in the interior of τ_m. Consecutive τ_m are adjacent. This implies that the path is one path component of $F^{-1}(0) \cap \mathbb{R}^2 \times (0,1]$

<u>Preliminary constructions</u>

Let

$$X' = \{z_n : z_n \text{ has prelabel } X\}$$

$$Y' = \{z_n : z_n \text{ has prelabel } Y\}$$

$$W' = \{z_n : z_n \text{ has prelabel } W\}.$$

Note that $\{z_n\}_{n=1}^{\infty} = X' \cup Y' \cup W'$ and this union is disjoint. Since $\{z_n\}_{n=1}^{\infty}$ is contained in the union of a properly nested sequence of bounded closed sets whose diameters go to 0 (namely the projections of $B(i,1)$, $i \geq 0$, into $\mathbb{R}^2 \times \{1\}$), there is a limit point z_0 of $\{z_n\}_{n=1}^{\infty}$. This implies that the path converges to $(z_0,0)$.

<u>Claim 1</u>: There is a line L in $\mathbb{R}^2$ through z_0 such that the perpendicular projections of $\{z_n\}_{n=0}^{\infty}$ onto L are all distinct.

<u>Proof</u>: $\{z_n\}_{n=0}^{\infty}$ is countable. The number of distinct slopes of line segments through pairs $\{z_i, z_j\}_{i \neq j}$ is at most countable. Choose a number m which is not the negative reciprocal of any of these slopes. Let L be the line through z_0 with slope m. ///

Let L' be the line in $\mathbb{R}^2$ perpendicular to L at z_0. Let L, L' be a second pair of coordinate axes of $\mathbb{R}^2$, with the same scale as the original axes. Let $\pi : \mathbb{R}^2 \to \mathbb{R}^1$ be the perpendicular projection map of $\mathbb{R}^2$ onto L, viewing L as a copy of $\mathbb{R}^1$. Let $\pi_n = \pi(z_n)$. Note $\pi_0 = \pi(z_0) = 0$.

A map $g : \mathbb{R}^n \to \mathbb{R}^1$ is of class C^0 if g is continuous. If all partial derivatives of order $\leq r$ exist and are continuous, then g is of class C^r. If g is of class C^r for all r, $1 \leq r < \infty$, then g is of class C^{∞}. A map $g : \mathbb{R}^n \to \mathbb{R}^m$ given by $g(z) = (g_1(z), \ldots, g_m(z))$ is of class C^r if each $g_i : \mathbb{R}^n \to \mathbb{R}^1$ is of class C^r, $0 \leq r \leq \infty$ (Spivak [9]).

Since π is a linear map on $\mathbb{R}^2$, it is of class C^∞. By the continuity of π, $\{\pi_n\}_{n=1}^\infty$ has the limit point π_0. Since the π_n, $n \geq 1$, are isolated, we can construct intervals $I_n = (\pi_n - \varepsilon_n/2, \pi_n + \varepsilon_n/2)$, where $0 < \varepsilon_n < 1$ is chosen so that the closures of the I_n are pairwise disjoint and don't contain π_0. For any set A, we denote the closure of A by $\bar{A}$.

Let $\phi : \mathbb{R}^1 \to [0,1]$ be a function of class C^∞ satisfying

1. $\phi \equiv 1$ on $[-1/6, 1/6]$
2. $\phi > 0$ on $(-1/2, 1/2)$
3. $\phi \equiv 0$ outside $[-1/2, 1/2]$.

Note that the support of ϕ, denoted by supp(ϕ), satisfies

$$\text{supp}(\phi) = \overline{\{x : \phi(x) \neq 0\}} = [-1/2, 1/2] .$$

One such ϕ can be constructed as follows: Let $f(x) = e^{-1/x}$, $x > 0$, and $f(x) = 0$ for $x \leq 0$; f is of class C^∞, and $f(x) > 0$ for $x > 0$. Let $g(x) = f(x)/(f(x) + f(1-x))$. Then g is of class C^∞ and satisfies $g(x) = 0$ for $x \leq 0$, $g'(x) > 0$ for $0 < x < 1$, and $g(x) = 1$ for $x \geq 1$. Finally, let $\phi(x) = g(3x + 3/2)\, g(-3x + 3/2)$. Then ϕ is of class C^∞ and satisfies $\phi(x) = 0$ for $|x| \geq 1/2$, $\phi(x) > 0$ for $|x| < 1/2$, and $\phi(x) = 1$ for $|x| \leq 1/6$ (Munkres [8]). See Figure 5.

Given an interval $J = (a,b)$, $-\infty < a < b < \infty$, let $\phi(J)$ denote the ϕ function scaled and translated so that $\text{supp}(\phi(J)) = \bar{J}$.

Explicitly,

$$\phi(J)(x) = \phi(2^{-1}(2x-a-b)/(b-a)) .$$

In particular, we have

$$\phi(I_n)(x) = \phi((x-\pi_n)/\varepsilon_n) .$$

Note that

$$(\phi(I_n))^{(j)}(x) = (1/\varepsilon_n)^j \, \phi^{(j)}((x-\pi_n)/\varepsilon_n),$$

where (j) denotes the j-th derivative with respect to x. The 0-th derivative is the function itself.

For any function $k : \mathbb{R}^1 \to \mathbb{R}^1$, let $\|k\| = \sup\{|k(x)| : x \in \mathbb{R}^1\}$. If k is continuous, we can also write $\|k\| = \sup\{|k(x)| : x \in \mathrm{supp}(k)\}$.

<u>Claim 2</u>: $\|\phi^{(j)}\| \leq \|\phi^{(j+1)}\|$, $j = 0, 1, 2, \ldots$

<u>Proof</u>: Note that $\mathrm{supp}(\phi^0) = \mathrm{supp}(\phi) = [-1/2,1/2]$, and that $\mathrm{supp}(\phi^{(j)}) \subseteq \mathrm{supp}(\phi)$ for all j. By the Fundamental Theorem of calculus,

$$\phi^{(j)}(x) = \int_{-1/2}^{x} \phi^{(j+1)}(t)\, dt .$$

$$|\phi^{(j)}(x)| \leq \int_{-1/2}^{x} |\phi^{(j+1)}(t)| dt \leq \|\phi^{(j+1)}\| .$$

Taking the supremum of $|\phi^{(j)}(x)|$ over $x \in \mathrm{supp}(\phi)$ gives $\|\phi^{(j)}\| \leq \|\phi^{(j+1)}\|$. ///

Lemma: Let $k : \mathbb{R}^1 \to \mathbb{R}^1$ be continuous everywhere, and differentiable for all $x \neq 0$. If $\lim_{x \to 0} k^{(1)}(x) = 0$, then k is differentiable at $x = 0$, and $k^{(1)}(x) = 0$.

Proof: For any $x \neq 0$, apply the mean value theorem to get $(k(x) - k(0))/x = k^{(1)}(c)$ for some c between 0 and x. Thus, $\lim_{x \to 0} (k(x) - k(0))/x = \lim_{c \to 0} k^{(1)}(c) = 0$. But this is exactly $k^{(1)}(0)$. ///

Our desired function $f : \mathbb{R}^2 \to \mathbb{R}^2$ will be given by $f(z) = (f_1(z), f_2(z))$, where $f_i : \mathbb{R}^2 \to \mathbb{R}^1$ is of class C^∞, $i = 1, 2$.

Construction of f_1

I shall construct a function $g_1 : \mathbb{R}^1 \to \mathbb{R}^1$ of class C^∞. Define f_1 by $f_1(z) = g_1(\pi(z))$. Note that compositions of functions of class C^∞ are of class C^∞. Define

$$S : \{z_n\}_{n=1}^{\infty} \to \{-1, 0, +1\} \text{ by}$$

$$s_n = S(z_n) = \begin{cases} +1 & \text{if } z_n \in X' \\ 0 & \text{if } z_n \in W' \\ -1 & \text{if } z_n \in Y' \end{cases}.$$

Let $\phi_n = \phi(I_n)$. Choose $\alpha_n > 0$ so that

$$\alpha_n \|\phi_n^{(n)}\| < 2^{-n}.$$

Note that

$$\alpha_n \|\phi_n^{(j)}\| = \alpha_n (1/\varepsilon_n)^j \|\phi^{(j)}\| \leq \alpha_n (1/\varepsilon_n)^n \|\phi^{(n)}\|$$

$$= \alpha_n \|\phi_n^{(n)}\| < 2^{-n} \quad \text{for} \quad j \leq n \;.$$

Define $g_1 : \mathbb{R}^1 \to \mathbb{R}^1$ by

$$g_1(x) = \sum_{n=1}^{\infty} s_n \, \alpha_n \, \phi_n(x) \;.$$

Since the supports of the ϕ_n do not overlap, this sum is well-defined. Note that $g_1(0) = 0$, and $g_1(x) = 0$ for all $x \notin \bigcup_{n=1}^{\infty} \ddot{I}_n$. Thus

$$|g_1(x)| \leq \|\alpha_n \ \phi_n\| < 2^{-n} \quad \text{for} \quad x \in \text{supp}(\phi_n) \;.$$

Since the intervals I_n *tend to the origin in* $\mathbb{R}^1$, we have that g_1 is of class C^0. See Figure 6.

<u>Claim 3</u>: g_1 is of class C^∞.

<u>Proof</u>: For $x \neq 0$, $g_1^{(1)}(x) = \sum_{n=1}^{\infty} s_n \, \alpha_n \, \phi_n^{(1)}(x)$. So $|g_1^{(1)}(x)| \leq \|\alpha_n \ \phi_n^{(1)}\| \leq \|\alpha_n \ \phi_n^{(n)}\| < 2^{-n}$ for $x \in \text{supp}(\phi_n)$. As above, $|g_1^{(1)}(x)| \to 0$ as $x \to 0$. By the lemma, $g_1^{(1)}(0)$ exists and equals 0. Thus, g_1 is of class C^1. Inductively, assume g_1 is of class C^r. As above, $|g_1^{(r+1)}(x)| \to 0$ as $x \to 0$. The lemma implies that g_1 is of class C^{r+1}. Thus, g_1 is of class C^∞. ///

Construction of f_2

Recall that the objective is to create a function of class C^∞ with a unique zero at z_0. I shall construct f_2 in two stages. First, I shall define a function $g_2 : \mathbb{R}^1 \to \mathbb{R}^1$ of class C^∞ satisfying

1. $g_2(\pi(z_n)) < 0$ for $z_n \in W'$
2. $g_2(\pi(z_n)) > 0$ for $z_n \in X' \cup Y'$
3. $g_2(\pi(z)) < 0$ whenever $g_1(\pi(z)) = 0$, $z \notin \pi^{-1}(0)$.

Then I shall define a function $h : \mathbb{R}^2 \to \mathbb{R}^1$ of class C^∞ satisfying

1. $h(z_n) = 0$, $n \geq 0$
2. $h(z) < 0$ for $z \in \pi^{-1}(0)$, $z \neq z_0$.

By defining $f_2(z) = g_2(\pi(z)) + h(z)$, we shall have the desired function $f(z) = (f_1(z), f_2(z))$.

For π_n corresponding to $z_n \in X' \cup Y'$, let $J_n = (\pi_n - \varepsilon_n/6, \pi_n + \varepsilon_n/6)$. For each pair J_{n_1}, J_{n_2} of two neighboring J_n intervals, with J_{n_1} lying to the left of J_{n_2}, and no other J_n intervals lying in between them, we let $K_{n_1} = (\pi_{n_1} + \varepsilon_{n_1}/3, \pi_{n_2} - \varepsilon_{n_2}/3)$. If J_{n_r} is the left-most J_n interval, and if J_{n_s} is the right-most J_n interval, let $K_{n_s} = (\pi_{n_r} - \varepsilon_{n_r}/3, \pi_{n_s} + \varepsilon_{n_s}/3)$. If there is some J_{n_p} for which no J_n interval lies between J_{n_p} and the origin, then let $K_{n_p} = (\pi_{n_p} + \varepsilon_{n_p}/3, 0)$ or $K_{n_p} = (0, \pi_{n_p} - \varepsilon_{n_p}/3)$, depending on whether J_{n_p} lies to the left or right of the origin respectively. Note that each $z_n \in W'$ lies outside K_{n_s} or lies in some K_{n_j}, $n_j \neq n_s$.

Let $\xi_n = \phi(J_n)$. Let $\psi_m = \phi(K_m)$ if $m \neq n_s$. Write $K_{n_s} = (a,b)$. Let $\psi_m = 1 - \phi((2a-b, 2b-a))$ for $m = n_s$. Choose $\beta_n > 0$ so that

$$\beta_n \|\xi_n^{(n)}\| < 2^{-n}$$

and choose $\nu_m > 0$ so that

$$\nu_m \|\psi_m^{(m)}\| < 2^{-m} .$$

Define

$$g_2(x) = \sum_n \beta_n \xi_n(x) - \sum_m \nu_m \psi_m(x) .$$

The first sum is over those indices for which we have defined a J_n interval. The second sum is over those indices for which we have defined a K_m interval. See Figure 7.

The proof that g_2 is of class C^∞ follows exactly as that for g_1. One caution is that we used the fact that $\text{diam}(I_n) < 1$. Here, either there are infinitely many K_m on both sides of the origin, whence $\text{diam}(K_m) < 1$ for m sufficiently large, or there are finitely many K_m, $m \neq n_s$, on one side of the origin. In this latter case, the conditions of the lemma are still satisfied since $\phi(K_{n_p})$ is of class C^∞.

Claim 4: For $z \notin \pi^{-1}(0)$, $g_1(\pi(z)) = 0$ implies $g_2(\pi(z)) < 0$.

Proof: Fix z, $z \notin \pi^{-1}(0)$. Then $g_1(\pi(z)) = 0$ implies $\pi(z) \notin I_n$ for any n corresponding to $z_n \in X' \cup Y'$. Either $\pi(z)$ lies outside K_{n_s}, or $\pi(z)$ lies in some K_m, $m \neq n_s$. In either case, $g_2(\pi(z)) = -\nu_m \psi_m(\pi(z)) < 0$. Note that $\text{supp}(\psi_{n_s}) = \{x : x \notin K_{n_s}\}$. ///

If we were to define $f(z) = (g_1(\pi(z)), g_2(\pi(z)))$, then $z_n \in X'$ implies $f(z_n) \in X$, $z_n \in Y'$ implies $f(z_n) \in Y$, $z_n \in W'$ implies $f(z_n) \in W$, and the prelabel requirements would be satisfied. But all $z \in \pi^{-1}(0)$ satisfy $g_1(\pi(z)) = g_2(\pi(z)) = 0$. In order that f have a unique zero at z_0, we modify this definition by the function h described above.

Construction of h

With no loss of generality, the constructions in this section will be with respect to the coordinate axes L and L'. Thus, $z_n \to (0,0)$ as $n \to \infty$, and no z_n, $n \geq 1$, lies on the L' axis. Note that the origin is the only cluster point of $\{z_n\}_{n=1}^{\infty}$. We will write $z = (x,y) \in L' \times L$.

Let $U_j = (2^{-(j+2)}, 2^{-j})$, $V_j = (-2^{-j}, -2^{-(j+2)})$; $(j \geq 1)$.

For each j, there is a $\delta_j > 0$ such that no z_n lies in

$$U_j \times (-\delta_j, \delta_j) \cup V_j \times (-\delta_j, \delta_j),$$

and $\delta_j < 2^{-j}$. Let $\tau_j = \phi(U_j) + \phi(V_j)$. Let $\sigma_j = \phi((-\delta_j/2, \delta_j/2))$.

Choose $\Delta_j > 0$ so that

$$\Delta_j \|\tau_j^{(j)}\| \|\sigma_j^{(j)}\| < 2^{-j}.$$

Define h by

$$h(z) = h((x,y)) = \sum_{j=1}^{\infty} -\Delta_j \tau_j(x)\, \sigma_j(y).$$

Since each z lies in at most two rectangles of the form

$$U_j \times (-\delta_j, \delta_j) \quad \text{or} \quad V_j \times (-\delta_j, \delta_j) ,$$

the sum is well-defined. Note that for each fixed $z \neq (0,0)$, there is some open neighborhood of z which intersects at most three rectangles of the form

$$U_j \times (-\delta_j, \delta_j) \quad \text{or} \quad V_j \times (-\delta_j, \delta_j)$$

with consecutive indices. So for some n ,

$$h(z) = \sum_{j=n}^{n+2} - \Delta_j \, \tau_j(x) \, \sigma_j(y) .$$

Note that $h((0,0)) = 0$.

Claim 5: h is continuous.

Proof: We need only establish continuity at $(0,0)$.

$$\begin{aligned} |h(z)| &= \left| \sum_{j=n}^{n+2} - \Delta_j \tau_j(x) \, \sigma_j(y) \right| \\ &\leq \sum_{j=n}^{n+2} \Delta_j \|\tau_j\| \, \|\sigma_j\| \\ &\leq \sum_{j=n}^{n+2} \Delta_j \|\tau_j^{(j)}\| \, \|\sigma_j^{(j)}\| \\ &< \sum_{j=n}^{n+2} 2^{-j} . \end{aligned}$$

As $z \to (0,0)$, $n \to \infty$, and $h(z) \to 0$. ///

__Claim 6__: h is of class C^{∞} .

__Proof__: $\frac{\partial h(z)}{\partial x} = \sum_{j=n}^{n+2} - \Delta_j \tau_j^{(1)}(x)\, \sigma_j(y)$ for $z \neq (0,0)$. This sum goes to zero as $z \to (0,0)$. Applying the lemma to the definition of the partial derivative gives $\frac{\partial h((0,0))}{\partial x}$ exists and equals 0 . Similarly, $\frac{\partial h(z)}{\partial y} \to 0$ as $z \to (0,0)$. Since

$$\frac{\partial h((0,0))}{\partial y} = \lim_{y \to 0} \frac{h((0,y)) - h((0,0))}{y} = 0 ,$$

both first order partial derivatives are continuous. Thus, h is of class C^1 .

Assuming h is of class C^r, we use the lemma to show that all partial derivatives of order $r + 1$ exist at $(0,0)$ and equal 0 . This, combined with the fact that all partial derivatives of order $r + 1$ approach 0 as $z \to (0,0)$, implies that h is of class C^{r+1} . By induction, h is of class C^{∞} . ///

The function h was needed to perturb the values of $g_2(\pi(z))$ for all $z \in \pi^{-1}(0)$, $z \neq z_0$. So far, we have only done this for a bounded portion of $\pi^{-1}(0)$. Recall that $\tau_1 = \phi(U_1) + \phi(V_1)$. Define $\hat{\tau}_1$ by

$$\hat{\tau}_1(x) = \begin{cases} \tau_1(x) & \text{for } |x| \leq 3/8 \\ 1 & \text{otherwise} \end{cases} .$$

Replace τ_1 by $\hat{\tau}_1$ in the definition of h . Now, $h(z) < 0$ for all $z \in \pi^{-1}(0)$, $z \neq z_0$, and, replacing δ_1 with a smaller value as needed, $h(z_n) = 0$, $n \geq 0$.

Degree of f at z_0

From our construction of f, we note that f does not map to any point in the set $\{(x,y) \in \mathbb{R}^2 : x = 0,\ y > 0\}$. We can, therefore, define arbitrarily small perturbations of f by

$$f_\varepsilon(z) = f(z) - (0,\varepsilon),\ \varepsilon > 0 ,$$

such that f_ε has no zero. The Eaves-Saigal algorithm still computes a path in $F_\varepsilon^{-1}(0) \cap \mathbb{R}^2 \times (0,1]$; F_ε being derived from f_ε. But this path will be greatly different from the one we constructed, even for small $\varepsilon > 0$. In particular, it cannot be converging.

The reason for this unstable behavior lies in the fact that the degree of f at z_0 is zero (Artin and Braun [1]). We easily compute this degree by noting that f is symmetric with respect to the line L. Thus, the image of any circle about z_0 *does not fully wrap* around the origin. This leads us to ask: Does there exist a function f satisfying

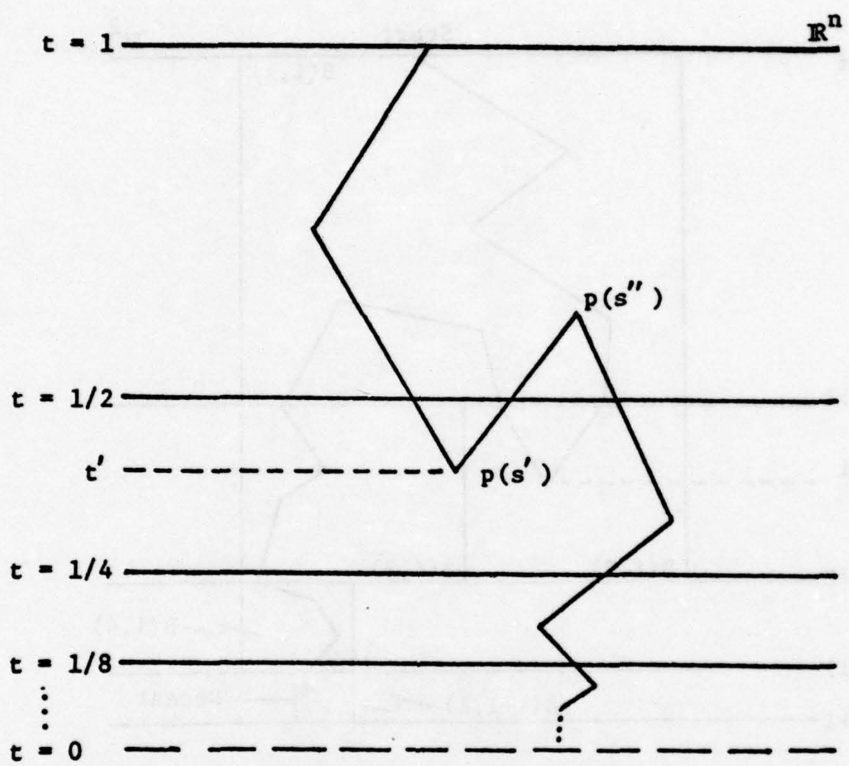
1. f is of class C^∞,
2. f has a unique zero at z_0,
3. $F^{-1}(0) \cap \mathbb{R}^n \times (0,1]$ has an infinitely retrogressing converging path component, and
4. the degree of f at z_0 is non-zero?

Acknowledgment

I wish to thank Professor B. C. Eaves for suggesting this problem, and for providing his insight and direction to our many discussions.

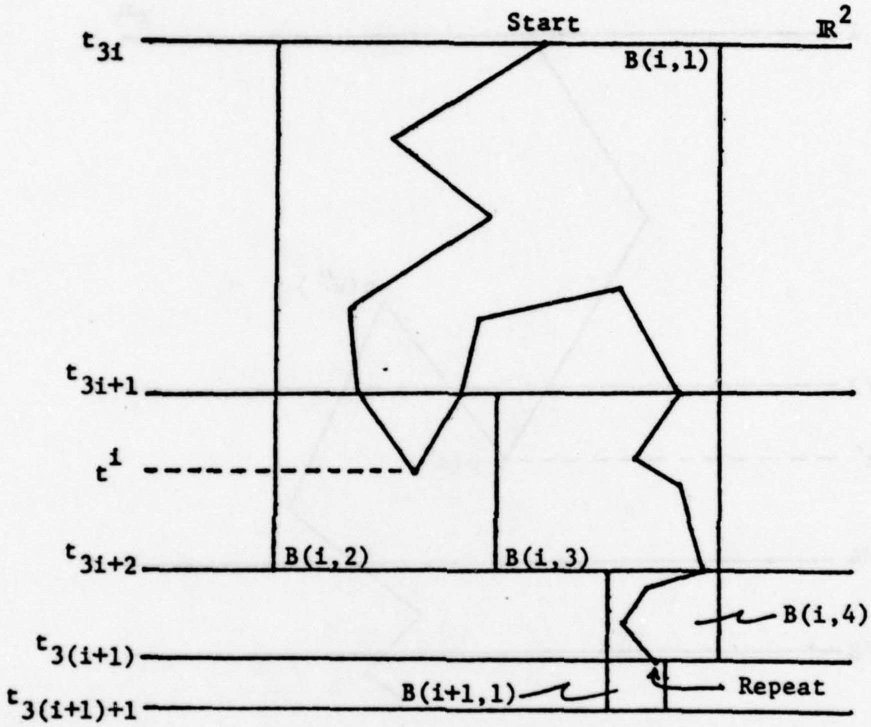
3/10/79

FIGURE 1



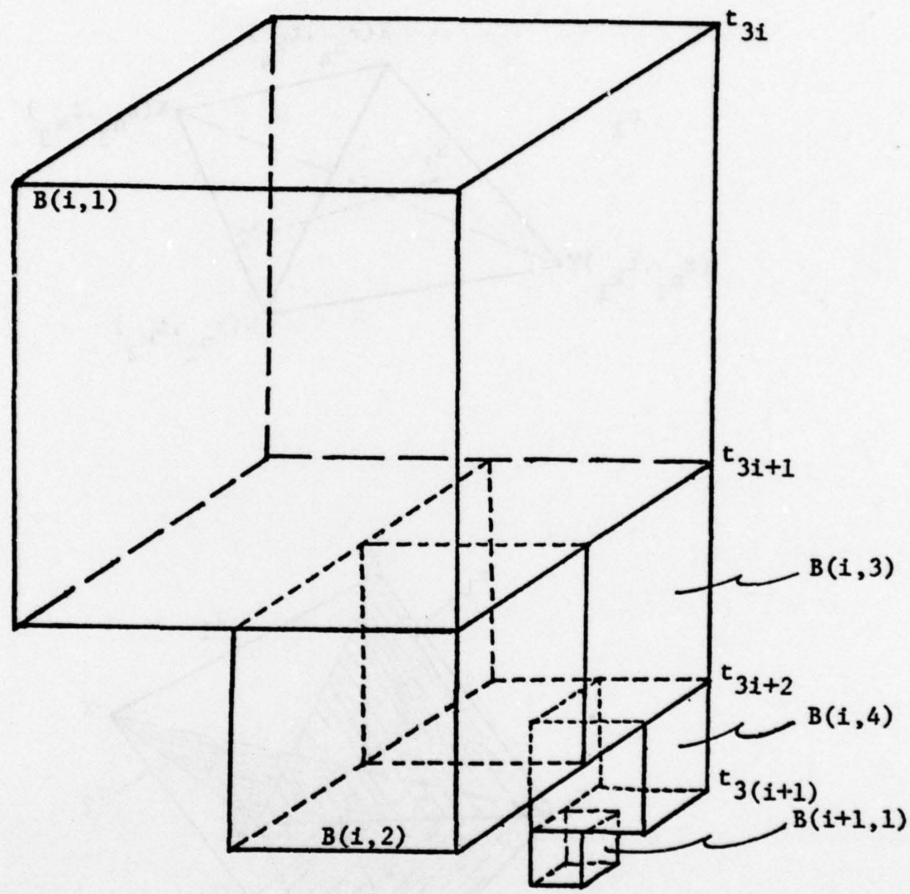
Retrogression of path p(s) at t′

FIGURE 2



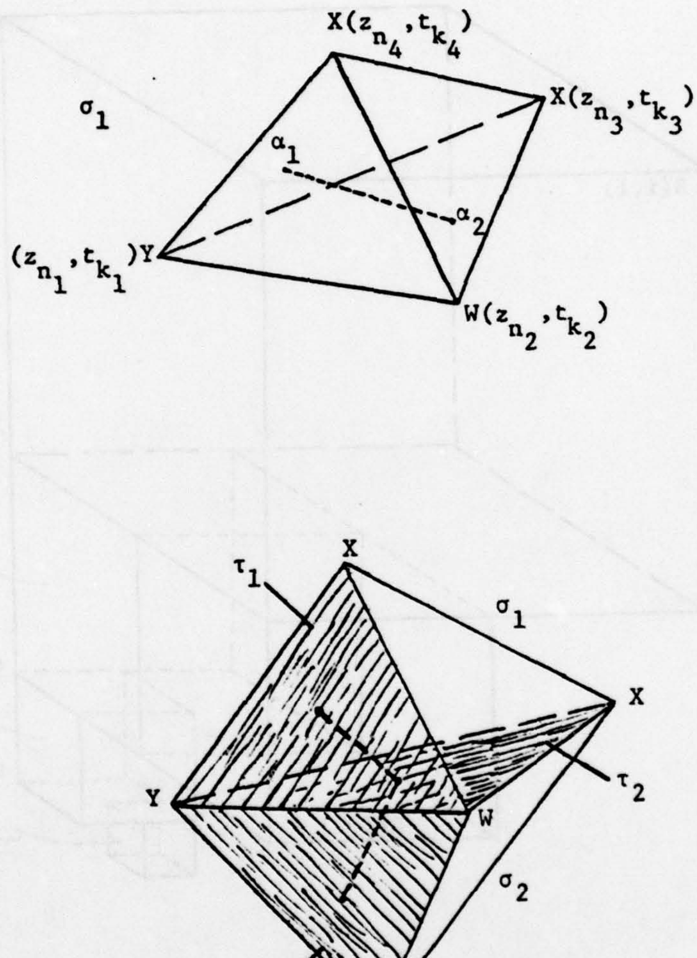
Schematic diagram of cycle i of path, with retrogression at t^i

FIGURE 3



Portion of $\mathbb{R}^2 \times (0,1]$ depicted in schematic diagram of Figure 2

FIGURE 4



$X(z_{n_4}, t_{k_4})$
$X(z_{n_3}, t_{k_3})$
σ_1
α_1
α_2
$(z_{n_1}, t_{k_1})Y$
$W(z_{n_2}, t_{k_2})$
X
τ_1
σ_1
X
τ_2
Y
W
σ_2
τ_3
$X(z_{n_5}, t_{k_5})$

FIGURE 5

Construction of ϕ

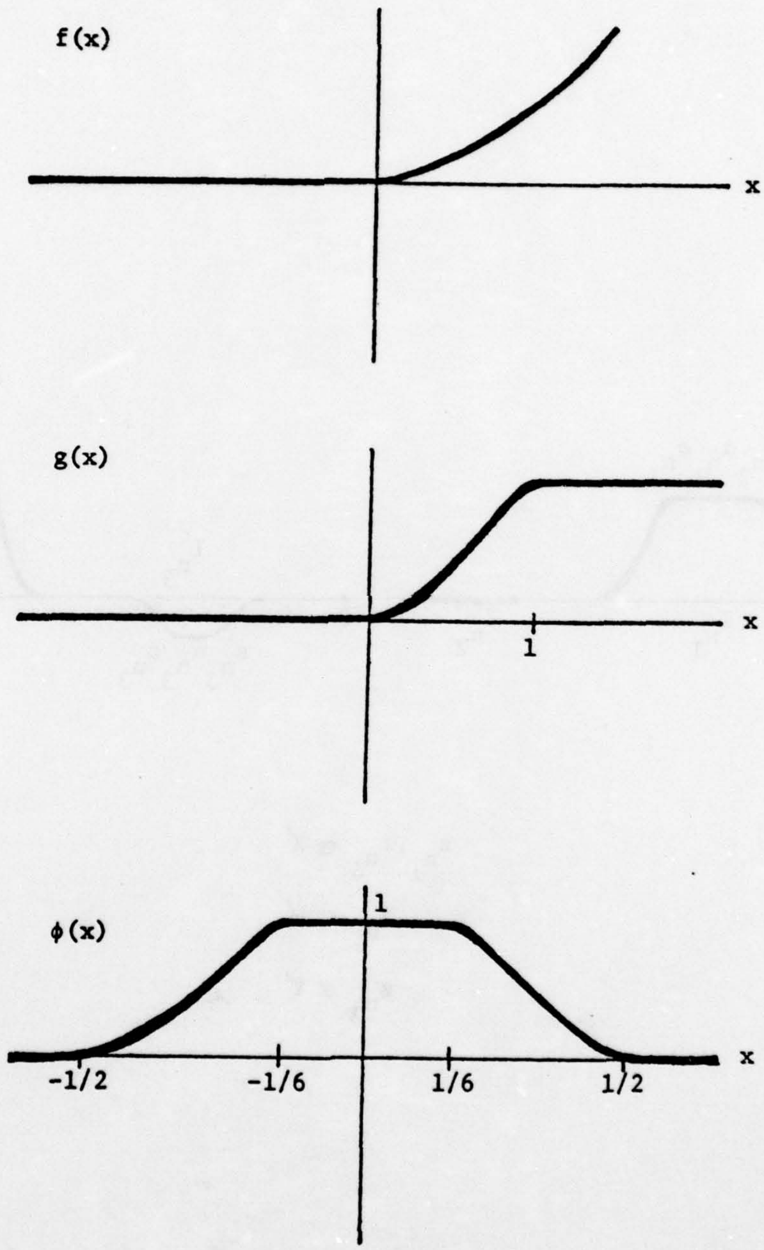
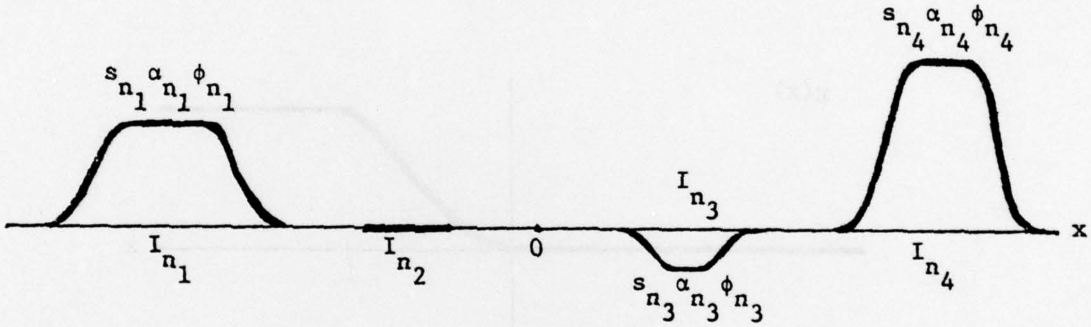


FIGURE 6

Illustration of the construction of g_1



$z_{n_1}, z_{n_4} \in X'$

$z_{n_2} \in W'$

$z_{n_3} \in Y'$

FIGURE 7

Illustration of the construction of g_2

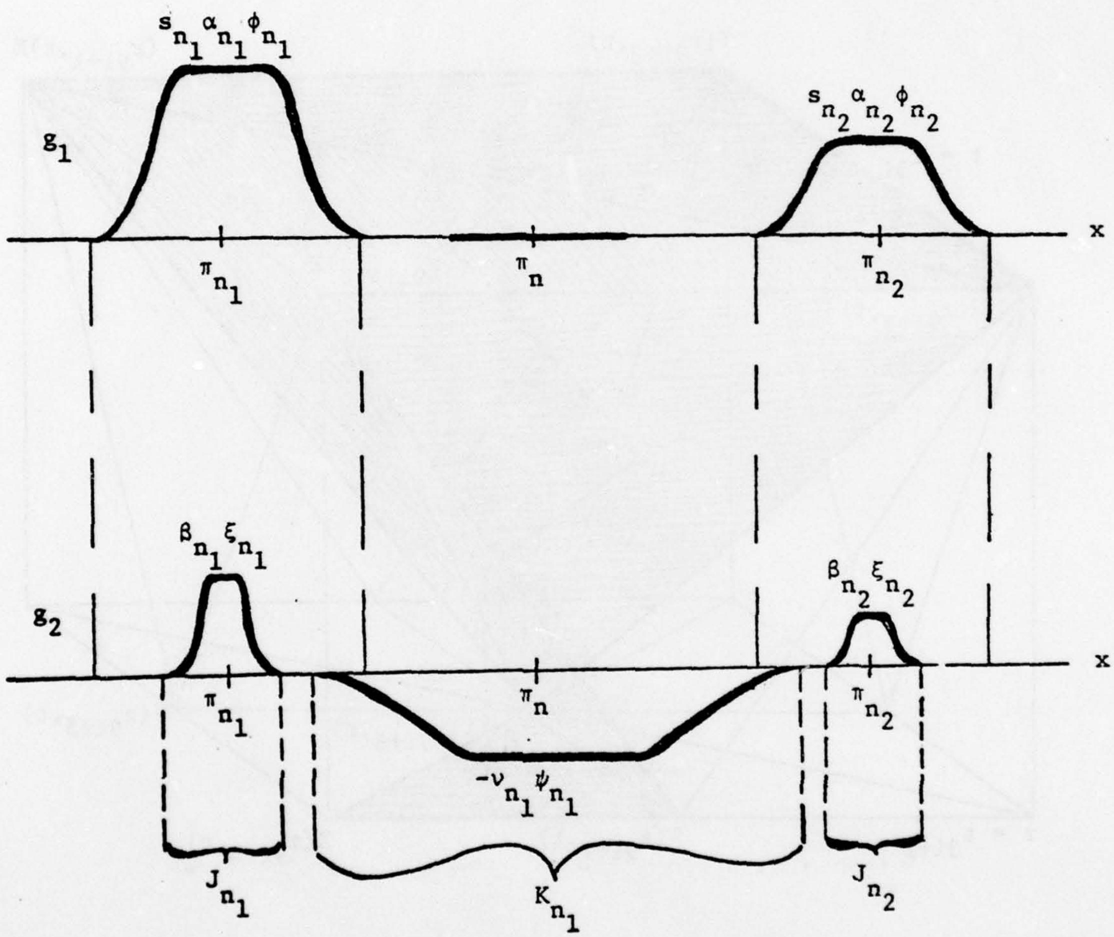


FIGURE 8

Block B(i,1)

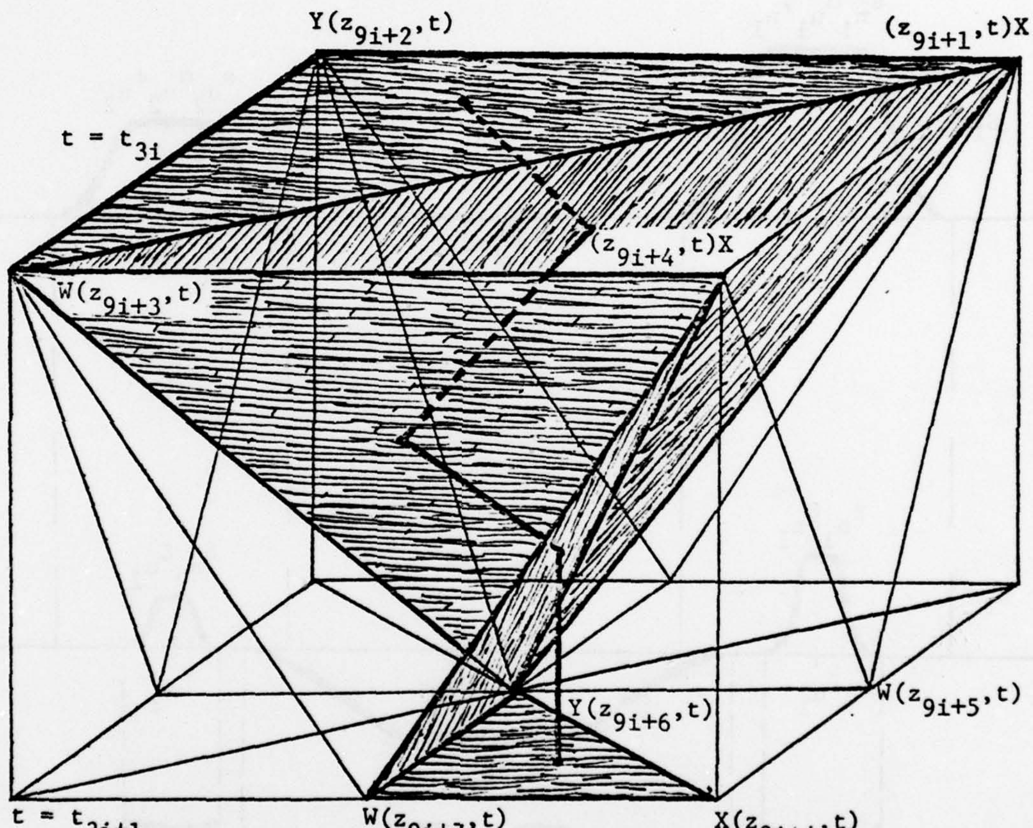


FIGURE 9

Block B(i,2)

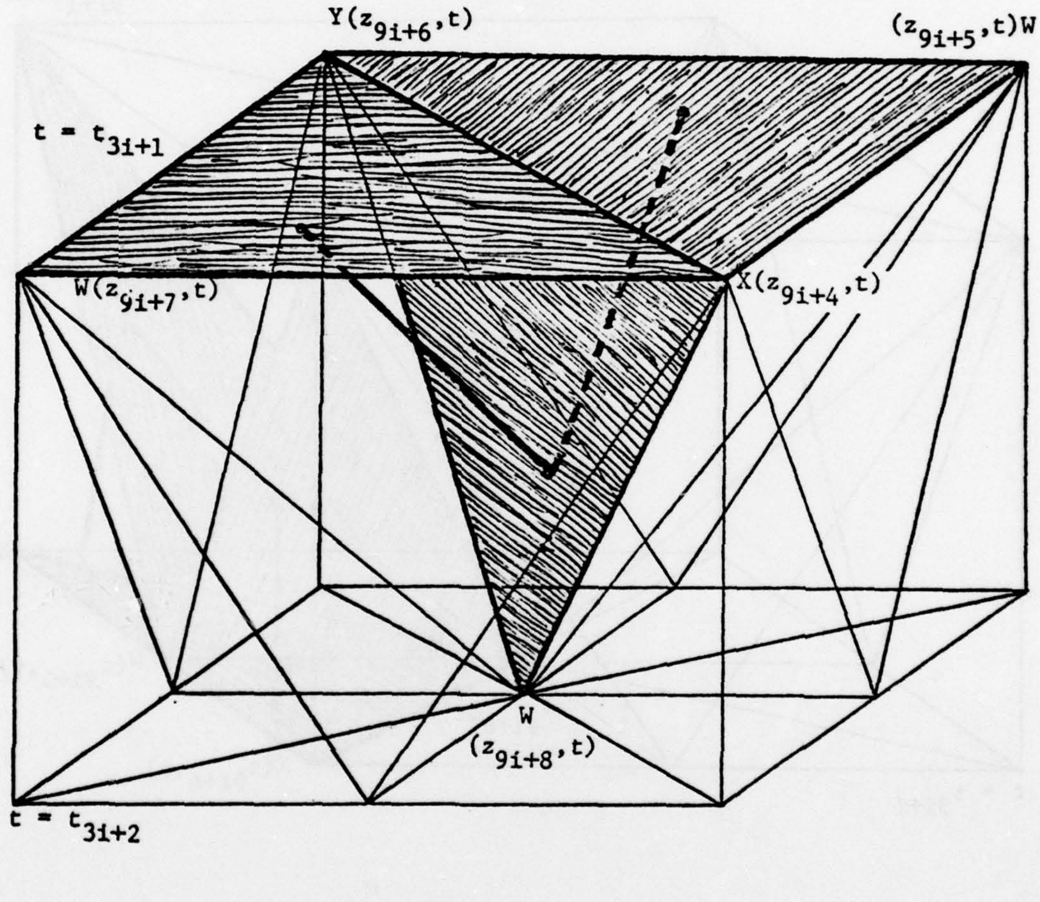


FIGURE 10

Block B(i,1)

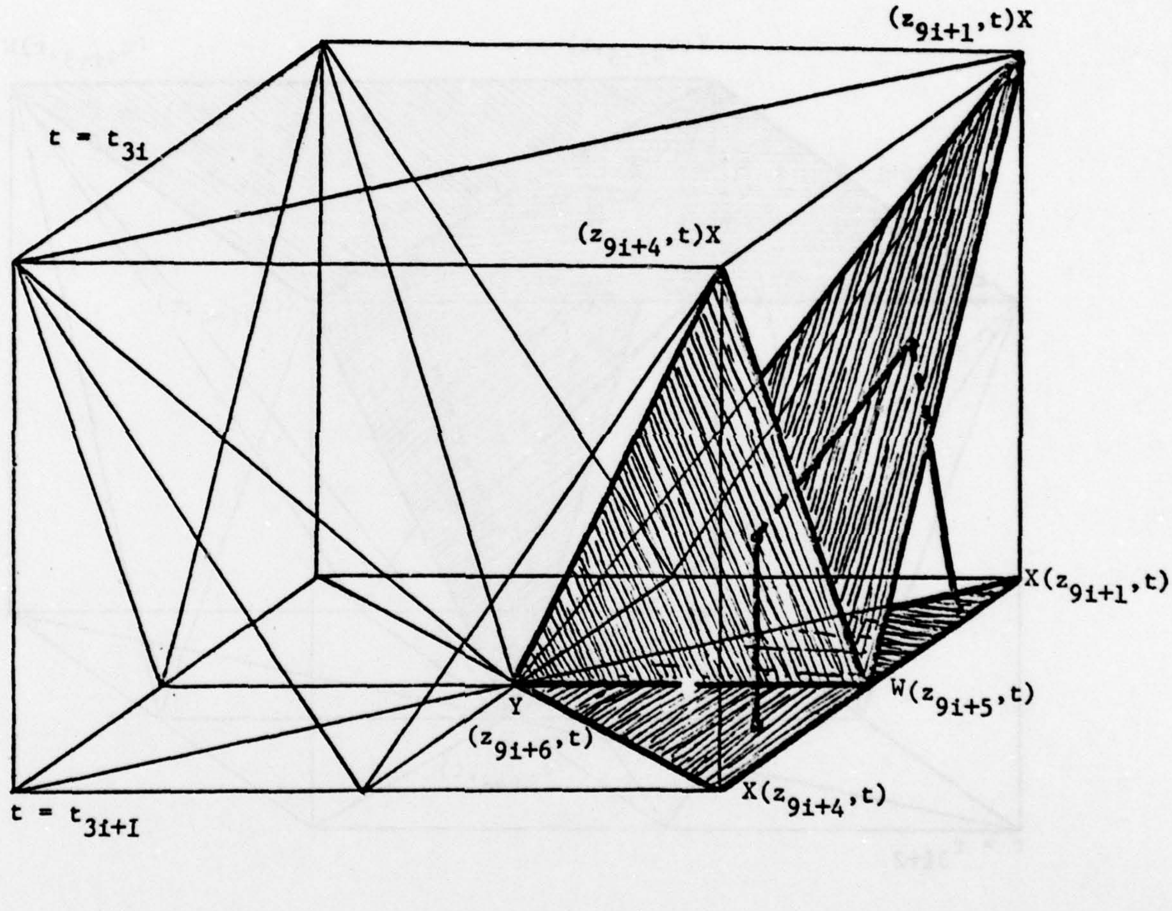


FIGURE 11

Block B(i,3)

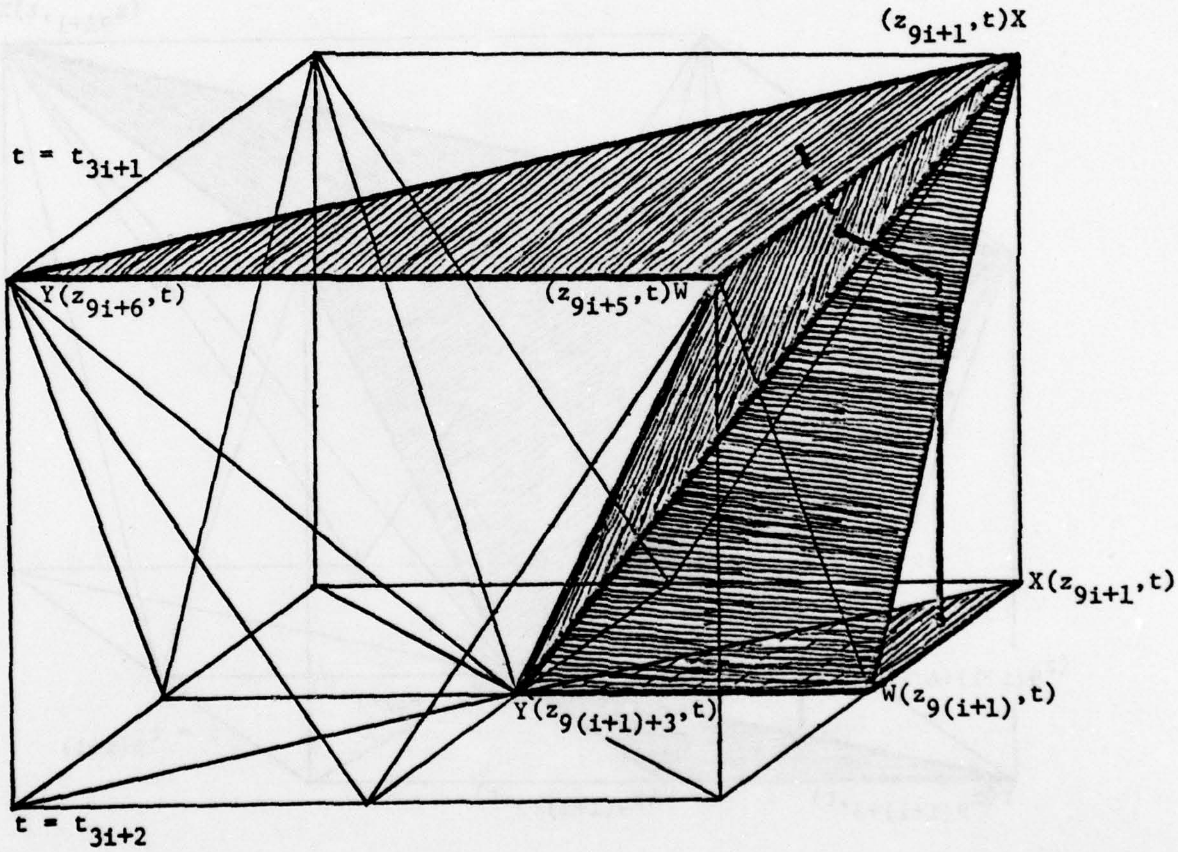
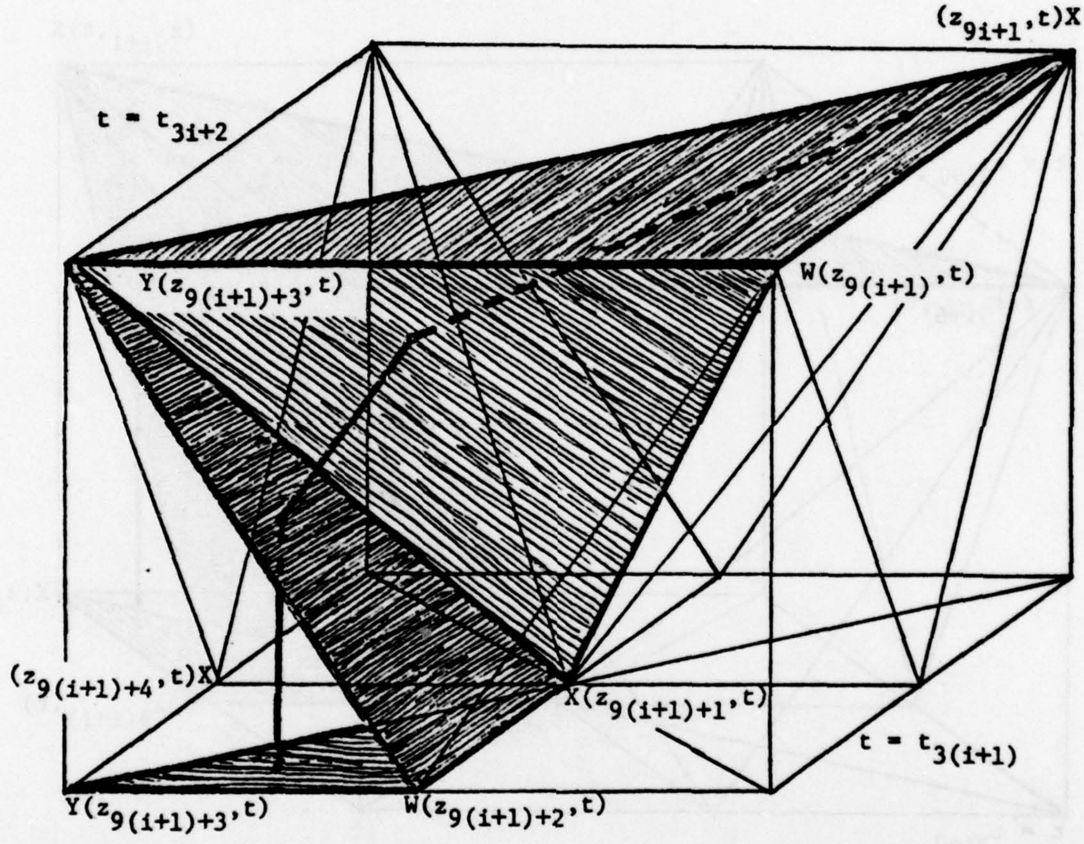


FIGURE 12

Block B(i,4)



REFERENCES

[1] ARTIN, E., and H. BRAUN, Introduction to Algebraic Topology, Charles E. Merrill Pub. Co., 1969.

[2] EAVES, B.C., "Computing Kakutani Fixed Points", SIAM J. of Appl. Math. 21 2 (1971), 236-244.

[3] ______, "Homotopies for Computation of Fixed Points", Math. Prog. 3 1 (1972), 1-22.

[4] ______, "Properly Labeled Simplexes", Studies in Optimization 10, MAA Studies in Mathematics, Ed: G.B. Dantzig and B.C. Eaves (1974), 71-93.

[5] ______, "A Short Course in Solving Equations with PL Homotopies", Nonlinear Programming, Proceedings of the Ninth SIAM-AMS Symposium in Applied Mathematics, Ed: R.W. Cottle and C.E. Lemke, SIAM, Philadelphia, 1976.

[6] EAVES, B.C., and R. SAIGAL, "Homotopies for Computation of Fixed Points on Unbounded Regions", Math. Prog. 3 2 (1972), 225-237.

[7] KOJIMA, M., "Studies in Piecewise-Linear Approximations of Piecewise-C^1 Mappings in Fixed Points and Complementarity Theory", Math. of Oper. Res. 3 1 (1978), 17-36.

[8] MUNKRES, J.R., Elementary Differential Topology, Annals of Mathematics Studies, Number 54, Princeton University Press, Princeton, New Jersey, 1966.

[9] SPIVAK, M., Calculus on Manifolds, W.A. Benjamin, Inc., 1965.

[10] TODD, M.J., "Union Jack Triangulations", in Fixed Points: Algorithms and Applications, Ed: Stephan Karamardian, Academic Press, 1975.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) AN INFINITELY RETROGRESSING CONVERGING PATH IN $F^{-1}(0)$ DERIVED FROM A C^{∞} FUNCTION AND THE J_3 TRIANGULATION		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Peter S. Brooks		8. CONTRACT OR GRANT NUMBER(s) DAAG-29-78-G-0026
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Operations Research, Stanford University, Stanford, California 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS U.S. Army Research Office, Box CM, Duke Station, Durham, N.C. 27706		12. REPORT DATE March 10, 1979
		13. NUMBER OF PAGES 30
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release and sale; its distribution is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

retrogressing paths
homotopy method
fixed-point algorithm

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For certain functions $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$, the Eaves-Saigal algorithm computes a path $p = (p_1,p_2) : [0,+\infty) \to F^{-1}(0) \cap \mathbb{R}^n \times (0,1]$, such that $(p_1(s), p_2(s)) \to (z,0)$ as $s \to +\infty$. It is shown that even when $F(\cdot,0)$ is of class C^{∞} and has a unique zero, $p_2(s)$ may not decrease monotonically to 0 on $[s_0, +\infty)$ for any s_0.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)